Up to now: Propositional Logic.

Predicate Logic and (Naive) Set Theory

Defn: A set is a collection of elements.

Notation: A set w/ (e.g.) elts a, b, c is denoted $\{a, b, c\} = S$.

We also say $a \in S$, $b \in S$, $c \in S$

i.e. "a is in S" or "a is an element of S"

Ex: $S = \{1, 2, 3, 4\}$

$T = \{1, 4, 9, 16, 25, 36, 37\}$ ← "roster notation"

NB: Roster notation only works for finite sets...

Q: How would we give an infinite set?

A: Use "set-builder notation"

Ex: "The set of all square natural numbers" looks like

$$\{x : x \text{ is a natural number and } x \text{ is a square}\}.$$

- x: variable symbol/description
- ":" "such that"
- "x is a natural number and x is a square": formal description of what x must satisfy. i.e. a predicate.

Defn: A *predicate* is a function taking variables (which can be specialized to elements of some set) to logical statements.

Ex: $P(x)$: "$x + 5 = 7$" is a predicate.
← applies to numbers

* $Q(x,y)$: "x Loves y" is a predicate.
← applies to people ←

$R(x)$: "x is the best." is **Not** a predicate.
↳ the truth value of $R(x)$ is indeterminate for any x...

Def: The *domain* of a predicate is the set of values allowed for substitution into the predicate.

Ex: $P(x)$: "$x + 5 = 7$" has domain $\{x : x \text{ is a real number}\}$.

NB: A predicate w/o a domain is ill-defined...
i.e. We absolutely need the domain of predicates we work with... Often we can intuit domains, but in formal mathematical statements, they **Must** be stated.

Ex: $T = \{x : x \text{ is a natural number and } x \text{ is a square}\}$

Let $N(x)$: "x is a natural number"

$S(x)$: "x is a square number"

So $T = \{x : N(x) \wedge S(x)\}$.

Point: We can build new predicates from old ones using the logical operations/connectives we studied...

NB: Given a predicate $P(x)$ (on a "universe" set), we can build a set: $S_P = \{x : P(x)\}$.

(in our example $*$: Domain $= \{x : x \text{ is a real number}\}$
predicate $= N(x) \wedge S(x)$
so our corresponding set is T)

On the other hand (= OTOH), given a set X, we can build a predicate $P_X(y)$: "$y \in X$".

In this way, we have a natural correspondence between set theory and predicate logic.

NB: The "universe set" can be thought of in two ways

① All the well-defined mathematical objects...

② "biggest collection of reasonable formulae".

→ Russel's Paradox. (set of all sets is no bueno).

Common Sets

$\mathbb{N} = \{x : x \text{ is a natural number}\} = \{0, 1, 2, \cdots\}$

$\mathbb{Z} = \{x : x \text{ is an integer}\} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$\mathbb{Z}_+ = \{x \in \mathbb{Z} : x > 0\} = \{1, 2, 3, \cdots\}$

$\mathbb{Q} = \{x : x \text{ is a rational number}\} = \{\frac{n}{d} : n \in \mathbb{Z}, d \in \mathbb{Z}, d \neq 0\}$

$\mathbb{R} = \{x : x \text{ is a real number}\}$

Pitfalls:

Defn: Two sets are equal when they have the same elements.

(i.e. $S = T$ iff every $x \in S$ satisfies $x \in T$ and every $x \in T$ satisfies $x \in S$).

Q: Does $\{1,2,3\} = \{3,2,1\}$?

yes! The order doesn't matter!!!

Q: Does $\{1,1\} = \{1\}$?

yes! duplicate elements don't matter!!!

Q: Does $\{\{1\},2\} = \{1,2\}$?

NO! $\{1\} \neq 1$

~~if $S \in S$ NONSENSE!~~

~~$S = \{S\} = \{\{S\}\} = \{\{\{S\}\}\} \ldots$~~

Quantification:

Statements like "Every car has four wheels" deserve to be translated as predicate statements...

i.e. "Every x which is a car has four wheels"

New Symbol: $\forall_{x \in D}[P(x)]$ reads as "For all x in D, $P(x)$ is true"

Ex: $F(x)$: "x has four wheels", $C = \{x : x \text{ is a car}\}$:

$\forall_{x \in C}[F(x)]$

$\forall$ is the universal quantifier.

Maybe we only know something satisfies the predicate...

e.g. "There is a truck with 18 wheels".

Introduce one more symbol: $\exists_{x \in D}[P(x)]$

reads "there exists an $x \in D$ satisfying $P(x)$".

e.g. Let $W(x)$: "x has 18 wheels"

and $T = \{x : x \text{ is a truck}\}$

$$\exists_{x \in T}[W(x)].$$

$\exists$ is the existential quantifier.

Quantified predicate statements are the language of maths ☺

Ex: In calculus 1, the definition of a limit is:

"Function f limits to $L \in \mathbb{R}$ at $a \in \mathbb{R}$ when for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x)-L| < \varepsilon$."

$P(f,a,L)$: "f limits to L at a".

$$P(f,a,L) \equiv \forall_{x \in \mathbb{R}} \forall_{\varepsilon > 0} \exists_{\delta > 0} [|x-a| < \delta \to |f(x)-L| < \varepsilon]$$

Set Operations and Definitions

Defn: Sets S and T are *equal* when

$$\forall x \left[x \in S \leftrightarrow x \in T \right].$$

(i.e. S and T have the same elements).

Defn: Set S is a *subset* of set T when every element of S is an element of T.
we write $S \subseteq T$

↑ not pointy...

Formally: $S \subseteq T$ iff $\forall x \left[x \in S \rightarrow x \in T \right]$.

Ex: $\{1, 3, 5\} \subseteq \mathbb{N}$.

$\{2, 5, 7, 11\} \subseteq \{x \in \mathbb{R} : x > 0\}$

$\{1, 3, 7\} \subseteq \{1, 2, 3, 5, 7, 11, 15\}$

$A = \{1, 3, 7\} \not\subseteq \{1, 3, 5, 9, 15\} = B$

$7 \in A$ but $7 \notin B$

Next Time: Set operations and proofs.

Intersection ~ and
Union ~ or
Set difference ~ "relative not"